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The discrete self-trapping equation and the Painlevé property

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Abstract. From the discrete self-trapping (DST) equation for three degrees of freedom rewritten in terms of the density matrix one can derive an autonomous system of real first-order ordinary differential equations by using SU(3) notation. Performing a Painlevé analysis, it is found that, depending on the parameters of the system, integrable and non-integrable cases can be distinguished.

1. Introduction

The discrete self-trapping (DST) equation

$$\left(i \frac{d}{dt} - \omega_0\right) \mathbf{A} + \Gamma \mathbf{D}(|\mathbf{A}|^2) \mathbf{A} + \varepsilon \mathbf{M} \mathbf{A} = 0 \quad (1.1)$$

was introduced by Eilbeck *et al* (1985) as a model to describe the nonlinear dynamics of small polyatomic chains such as water, ammonia, methane, acetylene and benzene, as well as of larger molecules such as acetanilide. Here \mathbf{A} is a complex n -component vector, the components of which represent the probability amplitude of finding some conserved quantity on the n th subunit of the structure. The diagonal matrix $\mathbf{D}(|\mathbf{A}|^2) = \text{diag}(|A_1|^2, |A_2|^2, \dots, |A_n|^2)$ which appears in the nonlinear term of (1.1) represents the tendency of \mathbf{A} to self-trap through a nonlinear interaction with the adjacent structure. The strength of this interaction is specified by the positive parameter Γ . \mathbf{M} is a real symmetric matrix ($m_{ij} = m_{ji}$) with zero diagonal elements.

The DST system is Hamiltonian and has the conserved quantity $N = \sum_{i=1}^n |A_i|^2$. It has two integrable limits, corresponding respectively to $\Gamma = 0$ and $\varepsilon = 0$. The properties of (1.1) and, in particular, the appearance of chaotic behaviour for $n > 2$ were investigated by Eilbeck *et al* (1985), Jensen *et al* (1985), De Filippo *et al* (1988) and Cruzeiro-Hansson *et al* (1990). While solutions for its continuum limit (the one-dimensional nonlinear Schrödinger equation) are well known (Scott *et al* 1973), analytical solutions for the discrete chain are not known except for the case $n = 2$.

We extend model (1.1) to the more general case when both the strength of the nonlinear interaction and the harmonic frequency depend on the site index n . This can be the case when different subunits constitute the chain. Therefore, we introduce in (1.1) the column vector

$$\Gamma = (\gamma_1, \gamma_2, \dots, \gamma_n) \quad (1.2)$$

of real components which takes into account anisotropic nonlinear coupling effects and

$$\Omega = (\omega_{01}, \omega_{02}, \dots, \omega_{0n}) \quad (1.3)$$

as the n -component vector for site-dependent frequencies.

In this way, we have included disorder in the site frequencies ω_{0i} , in the nonlinear interaction strengths γ_i and in the coupling matrix with elements m_{ij} .

In this paper we perform a Painlevé analysis for a system of first-order ordinary differential equations (ODEs) derived from the $n = 3$ DST equation rewritten in terms of the density matrix by using SU(3) notation. In particular, we look for all values of the parameters of the system such that its solutions have the Painlevé property, indicating integrability.

The paper is organized as follows. Section 2 is concerned with the Painlevé property and its relation to the integrability of a given system of ODEs. Section 3 is related to the DST equation with three degrees of freedom and its formulation as a real-valued Hamiltonian system of first-order ODEs. In section 4 we apply the Painlevé test to the system of ODEs related to the DST equation for the case $n = 3$. Finally, section 5 provides a brief summary of the results.

2. Integrability and the Painlevé property

Much attention has been focused on the question of whether a given dynamical system is integrable or non-integrable. Only a small number of integrable systems are known to date and it has been widely recognized that most (nonlinear) dynamical systems are non-integrable.

A direct method to identify integrable systems is the so-called singular point analysis (also called the Painlevé test). In this analysis the structure of singularities of the solution of ODEs is studied in the complex (time) plane. An ODE (or a system) in the complex domain is said to be of Painlevé type (or has the Painlevé property) if the only movable singularities of its solutions are poles. This means there are no moveable branch points or movable essential singularities (Ablowitz *et al* 1980).

There are some recent theorems which describe the connection between singular point analysis and integrability. Adler and van Moerbeke (1982a, b) and van Moerbeke (1988) proved for a class of Hamiltonian systems that passing the Painlevé test is a necessary condition for algebraic complete integrability in terms of Abelian function. Yoshida (1983a, b) proved that, if an autonomous system of first order ODEs does not pass the Painlevé test by a certain degree (i.e. has complex or irrational resonances), then the system cannot be algebraically integrable. An ODE (or a system) is said to be of the Painlevé type if all its solutions possess the Painlevé property.

A necessary condition that an autonomous system of ODEs

$$\frac{d\mathbf{w}}{dz} = \mathbf{g}(\mathbf{w}) \quad (2.1)$$

where \mathbf{g} is rational in $\mathbf{w} = (w_1, w_2, \dots, w_n)$, has the Painlevé property is that there is a Laurent expansion

$$w_i(z) = (z - z_1)^m \sum_{j=0}^{\infty} a_{ij}(z - z_1)^j \quad i = 1, 2, \dots, n \quad (2.2)$$

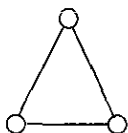
where $n - 1$ expansion coefficients are arbitrary (Steeb and Euler 1988). This necessary condition will be applied in the following.

3. The DST equation for $n = 3$

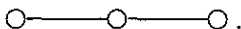
Now we consider the case of a trimer, which is the first nontrivial case beyond the dimer. The DST equation for $n = 2$ can easily be integrated. For $n = 3$ the DST system reads

$$i \frac{d}{dt} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = - \begin{bmatrix} \gamma_1 |A_1|^2 - \omega_{01} & m_{12} & m_{13} \\ m_{21} & \gamma_2 |A_2|^2 - \omega_{02} & m_{23} \\ m_{31} & m_{32} & \gamma_3 |A_3|^2 - \omega_{03} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}. \quad (3.1)$$

There exist two possible configurations, depending on the form of the linear exchange interaction: the triangular configuration, namely a closed chain of three sites where each site is coupled to its neighbouring site,



and the open chain where only one of the three sites is connected with the two others via a linear exchange interaction,



The system of three coupled equations (3.1) for the complex amplitudes A_i can be rewritten in terms of the density matrix ρ with elements $\rho_{jk} = A_j A_k^*$ using SU(3) notation and expanding ρ in terms of the corresponding generators.

We define the following real-valued variables through

$$u_i = A_j^* (\lambda_i)_{jk} A_k \quad i = 1, 2, \dots, 8 \quad (3.2)$$

where λ_i are the generators of the Lie group SU(3) for which the standard form Gell-Mann matrices are taken.

Equations (3.1) can now be expressed as a real-valued Hamiltonian system with a Lie-Poisson bracket defined on the dual space of the Lie algebra so(8) with dual coordinates $u = (u_1, \dots, u_8)$ and the bracket between two functions $F, G : \text{so}(8)^* \rightarrow \mathbb{R}$ is given by

$$\{F, G\} = \sum_{i,j,k=1}^8 c_{ij}^k u_k \frac{\partial F}{\partial u_i} \frac{\partial G}{\partial u_j} \quad (3.3)$$

where c_{ij}^k are the structure constants of so(8).

With respect to the Lie-Poisson bracket (3.3) the equations of motion may be written in Hamiltonian form as

$$\dot{u}_i = \sum_{j,k=1}^8 c_{ij}^k u_k \frac{\partial H}{\partial u_j} \quad i = 1, \dots, 8 \quad (3.4)$$

with the following Hamiltonian function

$$\begin{aligned} H = & -\frac{1}{24}(\gamma_1 + \gamma_2 + \gamma_3) \sum_{i=1}^8 u_i^2 - \frac{1}{8} \left\{ \frac{1}{3}(\gamma_1 + \gamma_2 + \gamma_3) u_8^2 + (\gamma_1 + \gamma_2) \frac{4}{3} N u_8 / (\sqrt{3}) \right. \\ & + (\gamma_1 - \gamma_2) \left[\frac{4}{3} N u_3 + (2/\sqrt{3}) u_3 u_8 \right] + \gamma_3 u_8^2 - \frac{8}{3} \gamma_3 N u_8 / (\sqrt{3}) \} \\ & + \frac{1}{2}(\omega_{01} + \omega_{02} - 2\omega_{03}) u_8 / \sqrt{3} + \frac{1}{2}(\omega_{01} - \omega_{02}) u_3 \\ & - m_{12} u_1 - m_{13} u_4 - m_{23} u_6. \end{aligned} \quad (3.5)$$

The square of the number N is a Casimir function for the Lie-Poisson bracket:

$$N^2 = \binom{8}{4} \sum_{i=1}^8 u_i^2. \tag{3.6}$$

The structure matrix of the Lie-Poisson manifold $\mathfrak{so}(8)^*$ with elements

$$J_{ij}(u) = \sum_{k=1}^8 c_{ij}^k u_k \quad u \in \mathfrak{so}(8)^*$$

is given by

$$\begin{bmatrix} 0 & 2u_3 & -2u_2 & u_7 & -u_6 & u_5 & -u_4 & 0 \\ -2u_3 & 0 & 2u_1 & u_6 & u_7 & -u_4 & -u_5 & 0 \\ 2u_2 & -2u_1 & 0 & u_5 & -u_4 & -u_7 & u_6 & 0 \\ -u_7 & -u_6 & -u_5 & 0 & \sqrt{3}u_8 + u_3 & u_2 & u_1 & -\sqrt{3}u_5 \\ u_6 & -u_7 & u_4 & -(\sqrt{3}u_8 + u_3) & 0 & -u_1 & u_2 & \sqrt{3}u_4 \\ -u_5 & u_4 & u_7 & -u_2 & u_1 & 0 & -(\sqrt{3}u_8 - u_3) & -\sqrt{3}u_7 \\ u_4 & u_5 & -u_6 & -u_1 & -u_2 & \sqrt{3}u_8 - u_3 & 0 & \sqrt{3}u_6 \\ 0 & 0 & 0 & \sqrt{3}u_5 & -\sqrt{3}u_4 & \sqrt{3}u_7 & -\sqrt{3}u_6 & 0 \end{bmatrix}. \tag{3.7}$$

In terms of the initial physical variables (the complex amplitudes A), the phase space is the six-real-dimensional product space $C \times C \times C$ possessing a Poisson bracket arising from a canonical symplectic form. In terms of the $SU(3)$ notation, the phase space is an eight-dimensional manifold defined on the dual space of the Lie algebra $\mathfrak{so}(8)$ and possesses a non-canonical Lie-Poisson structure. To cast system (3.4) into a more symmetric form we use, instead of the $SU(3)$ generators (the Gell-Mann matrices $\lambda_i, i = 1, 2, \dots, 8$), their following combinations:

$$\begin{aligned} F_1 = \lambda_1 & & F_2 = \lambda_2 & & F_3 = \lambda_3 & & F_4 = \lambda_4 & & F_5 = \lambda_5 \\ F_6 = \frac{1}{2}(\sqrt{3}\lambda_8 + \lambda_3) & & F_7 = \lambda_7 & & F_8 = \lambda_8 & & F_9 = \frac{1}{2}(\sqrt{3}\lambda_8 - \lambda_3) \end{aligned} \tag{3.8}$$

where each triple obeys the commutation relations

$$[F_i, F_j] = i \varepsilon_{ijk} F_k \quad i, j, k = \begin{bmatrix} 1, 2, 3 \\ 4, 5, 6 \\ 7, 8, 9 \end{bmatrix}.$$

Thus, each triple fulfils the Lie algebra $\mathfrak{su}(2) \approx \mathfrak{so}(3)$ and constitutes a subalgebra of the Lie algebra $\mathfrak{su}(3)$. Re-expressing the density matrix in terms of $F_i (i = 1, 2, \dots, 9)$ we find for the Hamiltonian function in the new variables u_i :

$$\begin{aligned} H = & -\frac{1}{24}(\gamma_1 + \gamma_2 + \gamma_3)[u_1^2 + u_2^2 + u_4^2 + u_5^2 + u_7^2 + u_8^2] \\ & -\frac{1}{18}\{\gamma, [2(u_3^2 + u_6^2) + u_3u_6 + 2(u_3 + u_6)N] + \gamma_2[2(u_3^2 + u_9^2) \\ & - u_3u_9 + 2(u_9 - u_3)N] + \gamma_3[2(u_6^2 + u_9^2) + u_6u_9 - 2(u_6 + u_9)N]\} \\ & - m_{12}u_1 - m_{13}u_4 - m_{23}u_7 \\ & + \frac{1}{3}[(\omega_{01} - \omega_{02})u_3 + (\omega_{01} - \omega_{03})u_6 + (\omega_{02} - \omega_{03})u_9]. \end{aligned} \tag{3.9}$$

The corresponding equations of motion may be expressed in Hamiltonian form,

$$\dot{u} = J(u) \frac{\partial H}{\partial u}$$

where $J(\mathbf{u})$ is a skew symmetric matrix with polynomial entries in \mathbf{u} , such that the Poisson bracket

$$\{F, G\} := \left\langle J \frac{\partial F}{\partial \mathbf{u}}, \frac{\partial G}{\partial \mathbf{u}} \right\rangle$$

satisfies the Jacobi identity.

The matrix $J(\mathbf{u})$ in block form is given by

$$J = \begin{bmatrix} S_1 & M_1 & -M_2^T \\ -M_1^T & S_4 & M_3 \\ M_2 & -M_3^T & S_7 \end{bmatrix} \quad S_i = \begin{bmatrix} 0 & u_{i+2} & -u_{i+1} \\ -u_{i+2} & 0 & u_i \\ u_{i+1} & -u_i & 0 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} u_8 & -u_7 & -u_2 \\ u_7 & u_8 & u_1 \\ u_5 & -u_4 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} -u_5 & u_4 & u_8 \\ u_4 & u_5 & -u_7 \\ -u_2 & u_1 & 0 \end{bmatrix} \quad M_3 = \begin{bmatrix} u_2 & u_1 & -u_5 \\ -u_1 & u_2 & u_4 \\ u_8 & -u_7 & 0 \end{bmatrix}. \tag{3.10}$$

The equation of motion in components are

$$\begin{aligned} \dot{u}_1 &= \frac{1}{2}(\gamma_1 + \gamma_2)u_2u_3 + \frac{1}{3}(\gamma_1 - \gamma_2)[N + \frac{1}{2}(u_6 + u_9)]u_2 - (\omega_{01} - \omega_{02})u_2 \\ &\quad - m_{23}u_5 - m_{13}u_8 \\ \dot{u}_2 &= -\frac{1}{2}(\gamma_1 + \gamma_2)u_1u_3 - \frac{1}{3}(\gamma_1 - \gamma_2)[N + \frac{1}{2}(u_6 + u_9)]u_1 + (\omega_{01} - \omega_{02})u_1 \\ &\quad + 2m_{12}u_3 + m_{23}u_4 - m_{13}u_7 \\ \dot{u}_3 &= -2m_{12}u_2 - m_{13}u_5 + m_{23}u_8 \\ \dot{u}_4 &= \frac{1}{2}(\gamma_1 + \gamma_3)u_5u_6 + \frac{1}{3}(\gamma_1 - \gamma_3)[N + \frac{1}{2}(u_3 - u_9)]u_5 - (\omega_{01} - \omega_{03})u_5 \\ &\quad - m_{32}u_2 + m_{12}u_8 \\ \dot{u}_5 &= -\frac{1}{2}(\gamma_1 + \gamma_3)u_4u_6 - \frac{1}{3}(\gamma_1 - \gamma_3)[N + \frac{1}{2}(u_3 - u_9)]u_4 + (\omega_{01} - \omega_{03})u_4 \\ &\quad + m_{32}u_1 + 2m_{13}u_6 - m_{12}u_7 \\ \dot{u}_6 &= -m_{12}u_2 - 2m_{31}u_5 - m_{23}u_8 \\ \dot{u}_7 &= \frac{1}{2}(\gamma_2 + \gamma_3)u_8u_9 + \frac{1}{3}(\gamma_2 - \gamma_3)[N - \frac{1}{2}(u_3 + u_6)]u_8 - (\omega_{02} - \omega_{03})u_8 \\ &\quad + m_{31}u_2 + m_{21}u_5 \\ \dot{u}_8 &= -\frac{1}{2}(\gamma_2 + \gamma_3)u_7u_9 - \frac{1}{2}(\gamma_2 - \gamma_3)[N - \frac{1}{2}(u_3 + u_6)]u_7 + (\omega_{02} - \omega_{03})u_7 + m_{31}u_1 \\ &\quad + 2m_{32}u_9 - m_{21}u_4 \\ \dot{u}_9 &= m_{12}u_2 - m_{31}u_5 - 2m_{23}u_8. \end{aligned} \tag{3.11}$$

There exists the trivial first integral

$$u_3 - u_6 + u_9 = 0. \tag{3.12}$$

Notice that in the special case of a transfer matrix with only one non-vanishing element (e.g. $m_{12} \neq 0$, $m_{13} = m_{23} = 0$ and the corresponding permutation of site indices) three of the equations decouple from the system to result in the system for an isolated dimer.

Motivated by a known mapping between an integrable nonlinear lattice Schrödinger equation and an integrable classical spin system (Ishimori 1982), we have studied this question for the $n = 3$ DST equation. We have found out that such an attempt fails.

Now we perform a Painlevé analysis of system (3.11). The quantities u_j are considered in the complex domain $u_j \rightarrow w_j = w_{j1} + iw_{j2}$. We are interested in identifying those sets of adjustable parameter values for which system (3.11) is integrable. In the next section these parameter sets are classified and the corresponding Painlevé test is performed for special situations.

4. Painlevé analysis for special nonlinear trimer configurations

In view of possible applications in trimer molecular aggregates we consider the following special cases depending on the explicit form of the nonlinear terms.

4.1. Completely anisotropic nonlinear interaction strengths ($\gamma_1 \neq \gamma_2 \neq \gamma_3$)

According to the first trivial integral (3.12) we substitute $w_9 = w_6 - w_3$ to obtain the reduced system in w_i ($i = 1, 2, \dots, 8$). First, we determine the dominant behaviour. Inserting the ansatz

$$w_i(z) \sim a_{i0}(z - z_1)^{k_i} \quad a_{i0} \neq 0$$

into (3.11), we calculate the k_j ($j = 1, 2, \dots, 8$) for which there is a balance of the leading terms. One finds the following two cases:

Case 1: $k_i = -1 \quad i = 3, 6, 7, 8 \quad k_j = -2 \quad j = 1, 2, 4, 5 \quad (4.1a)$

$$a_{10} = (2\gamma_3/\bar{\gamma})/m_{12} \quad a_{20} = \pm ia_{10} \quad a_{30} = \pm 2i(\gamma_2 + 2\gamma_3)/\bar{\gamma}$$

$$a_{40} = (2\gamma_2/\bar{\gamma})/m_{13} \quad a_{50} = \pm ia_{40} \quad a_{60} = \pm 2i(2\gamma_2 + \gamma_3)/\bar{\gamma}$$

$$a_{70} = \mp(2i/\bar{\gamma})/[(m_{13}/m_{12})\gamma_3 + (m_{12}/m_{13})\gamma_2]$$

$$a_{80} = (2/\bar{\gamma})[(m_{12}/m_{13})\gamma_2 - (m_{13}/m_{12})\gamma_3]$$

$$\bar{\gamma} = \gamma_1\gamma_2 + \gamma_1\gamma_3 + \gamma_2\gamma_3 \quad \gamma_2 \neq 0 \quad \gamma_3 \neq 0. \quad (4.1b)$$

Case 2: $k_i = -1 \quad i = 1, 2, 3, 4, 5, 6 \quad k_j = -2 \quad j = 7, 8. \quad (4.2a)$

From the system with the dominant behaviour we find for the two linear equations $\dot{w}_3 = -\dot{w}_6 = m_{23}w_8$. This leads to an asymmetry in the expansion coefficients $a_{30} = -a_{60}$ which is responsible for a remove of all γ_1 -dependent terms from the right-hand sides of the equations for the expansion coefficients a_{i0} with $i = 1, 2, 4, 5$. For the expansion coefficients a_{i0} one obtains

$$a_{10} = \pm 2i(m_{13}/m_{23})/(\gamma_3 - \gamma_2) \quad a_{20} = \mp ia_{10} \quad a_{30} = \mp 2i/(\gamma_2 + \gamma_3)$$

$$a_{40} = \pm 2i(m_{12}/m_{23})/(\gamma_3 - \gamma_2) \quad a_{50} = \mp ia_{40} \quad a_{60} = -a_{30} \quad (4.2b)$$

$$a_{70} = (2/m_{23})/(\gamma_2 + \gamma_3) \quad a_{80} = \pm ia_{70} \quad \gamma_2 \neq \gamma_3.$$

Next, we determine the resonances, i.e. the power of $(z - z_1)$ for which arbitrary constants may enter the series solution

$$w_i(z) = a_{i0}\tau^{k_i} + \sum_{j=1}^{\infty} a_{ij}\tau^{k_i+j} \quad \tau = (z - z_1).$$

We substitute

$$w_i(z) = a_{i0}\tau^{k_i} + b_i\tau^{k_i+r}$$

into the simplified equations of (3.11), which retain only the dominant terms, and find after a little algebra for the resonances:

$$\begin{aligned} \text{Case 1:} \quad & r = -1, 1(\text{twofold}), 2, 4(\text{twofold}) \\ & \text{and } r = \frac{1}{2} \pm \sqrt{\frac{1}{4} + (2\gamma_2\gamma_3)/\bar{\gamma}} \end{aligned} \tag{4.3}$$

$$\begin{aligned} \text{Case 2:} \quad & r = -1, 1, 2, 4, r = 1 \pm (2\gamma_2)/(\gamma_2 + \gamma_3) \\ & \text{and } r = 1 \pm (2\gamma_3)/(\gamma_2 + \gamma_3). \end{aligned} \tag{4.4}$$

One root is always -1 and is associated with the arbitrariness of the pole position at z_1 .

In order to determine those values of $\gamma_1, \gamma_2, \gamma_3$ for which the Painlevé property is satisfied, we require that all resonances, for both case 1 and case 2, are integers.

To get integer resonances for case 1 we require $(\gamma_2\gamma_3)/\bar{\gamma} = m(m-1)/2$ with $|m| > 1$ an integer in the last expression of (4.3). But this leads to an additional negative resonance $r_- < -1$ which must be ignored. From an expansion about a singularity one gets a seven-parameter solution and, hence, not the general eight-parameter form of the solution.

It is seen from the last two expressions of (4.4) that for all case 2 resonances to be integer a necessary condition is $\gamma_2 = \gamma_3$, which has to be excluded, however, due to (4.2b). Since not all resonances are integer the system does not pass the Painlevé test and is therefore non-integrable in this case.

4.2. Incomplete isotropic nonlinear interaction strengths ($\gamma_2 = \gamma_3 = \gamma, \gamma_1 \neq \gamma$)

The case of incomplete anisotropy in the nonlinear interaction strengths ($\gamma_2 = \gamma_3 = \gamma, \gamma_1 \neq \gamma$) is included in the results of case 1 of the previous section. Setting $\gamma_2 = \gamma_3 = \gamma$ in (4.3) one obtains for the resonances

$$\begin{aligned} & r = -1, 1(\text{twofold}), 2, 4(\text{twofold}) \\ & \text{and } r = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1/[1 + \frac{1}{2}(\gamma_1/\gamma)]}. \end{aligned} \tag{4.5}$$

To get only integers from the last expression of (4.5) we require $1/(1 + \frac{1}{2}(\gamma_1/\gamma)) = m(m-1)$ beneath the square root with $|m| > 1$ an integer. But this leads to an additional negative resonance $r_- < -1$ which must be ignored. Thus, the corresponding singular expansion is not the general solution.

4.3. Isotropic nonlinear interaction strength ($\gamma_1 = \gamma_2 = \gamma_3$)

It is easily seen that the results for the Painlevé test for a trimer with equal anharmonic parameters are included in the results of case 1 of section 4.1 as the limit case $\gamma_1 = \gamma_2 = \gamma_3$. The resonances are given by

$$r = -1, 1 \text{ (twofold)}, 2, 4 \text{ (twofold)} \text{ and } r = \frac{1}{2} \pm \sqrt{\frac{11}{12}}. \tag{4.6}$$

Two of the resonances are irrational, hence system (3.11) with equal nonlinearity parameters does not have the Painlevé property and we can apply Yoshida's theorem (1983) to show that the system is non-integrable. Numerical studies indicate that this system exhibits chaotic behaviour (Eilbeck *et al* 1985).

4.4. The anisotropic case of a two-site anharmonicity

The limit case of a two non-zero and one zero anharmonic parameter is also included in the general case of section 4.1. Setting $\gamma_1 = 0$ in (4.3) for the case 1 resonances one obtains

$$r = -1, 1, 2, 4 \tag{4.7}$$

and all resonances are doubly degenerated. Only six resonances are positive integers. This implies that the corresponding singular expansions are not generic. Since the whole of case 2 of section 4.1 is independent of the value of γ_1 , and even a vanishing γ_1 is allowed, the resonances for the two-site anharmonicity system with dominant behaviour (4.2a) are again given by (4.4); hence, the system does not possess the Painlevé property.

4.5. The case of a single-site anharmonicity

The case of a single-site anharmonicity (e.g. $\gamma_1 = \gamma_2 = 0, \gamma_3 \neq 0$) is included in case 2 of section 4.1. For the resonances one now gets

$$r = -1(\text{twofold}), 1(\text{twofold}), 2, 3, 4. \tag{4.8}$$

Thus, only a seven-parameter singular expansion is possible.

However, the case of only one non-vanishing nonlinearity parameter is not included in case 1 of section 4.1. This is due to the vanishing of $\tilde{\gamma} = \gamma_1\gamma_2 + \gamma_1\gamma_3 + \gamma_2\gamma_3$. System (3.11) now contains nonlinearities in only four of the eight equations whereas, in the presence of at least two non-vanishing nonlinearities, six equations were nonlinear in the previous cases.

To find all other possibilities for the dominant behaviour, it turns out that it is now more convenient to work with the system in SU(3) coordinates. Setting in (3.5) $\gamma_1 = \gamma_2 = 0$, one obtains with the help of (3.4) and (3.7) a system of eight coupled equations, with only four of them containing nonlinear terms connected with the anharmonic parameter γ_3 .

$$\begin{aligned} \dot{u}_1 &= -(\omega_{01} - \omega_{02})u_2 - m_{23}u_5 - m_{13}u_7 \\ \dot{u}_2 &= (\omega_{01} - \omega_{02})u_1 + 2m_{12}u_3 + m_{23}u_4 - m_{13}u_6 \\ \dot{u}_3 &= -2m_{12}u_2 - m_{13}u_5 + m_{23}u_7 & \dot{u}_4 &= \Delta_1(u_8)u_5 - m_{23}u_2 + m_{12}u_7 \\ \dot{u}_5 &= -\Delta_1(u_8)u_4 + m_{13}(u_3 + u_8) - m_{12}u_6 + m_{23}u_1 \\ \Delta_1(u_8) &= \frac{1}{3}\gamma_3(N + u_8) + \omega_{03} - \omega_{01} \\ \dot{u}_6 &= \Delta_2(u_8)u_7 + m_{13}u_2 + m_{12}u_5 \\ \dot{u}_7 &= -\Delta_2(u_8)u_6 + m_{31}u_1 - m_{21}u_4 + m_{32}(u_8 - u_5) \\ \Delta_2(u_8) &= \frac{1}{3}\gamma_3(N + u_8) + \omega_{03} - \omega_{02} & \dot{u}_8 &= -3m_{31}u_5 - 3m_{32}u_7. \end{aligned} \tag{4.9}$$

For a Painlevé test we regard these equations in the complex domain and first determine the dominant behaviour. Setting $u_j \rightarrow w_j = w_{j1} + iw_{j2}$ and inserting the ansatz

$$w_i(z) = a_{i0}(z - z_1)^{k_i}$$

in (4.9) we find

$$k_1 = k_2 = k_3 = k_8 = -1 \quad k_4 = k_5 = k_6 = k_7 = -2 \quad (4.10a)$$

and for the expansion coefficients

$$\begin{aligned} a_{10} &= \pm i(m_{23}a_{40} + m_{13}a_{60}) & a_{20} &= -m_{23}a_{40} + m_{13}a_{60} \\ a_{30} &= \pm i(m_{13}a_{40} - m_{23}a_{60}) & 2/\gamma_3 &= m_{31}a_{40} + m_{23}a_{60} \\ a_{50} &= \pm ia_{40} & a_{70} &= \pm ia_{60} & a_{80} &= \pm 6i/\gamma_3 \end{aligned} \quad (4.10b)$$

where a_{40} or a_{60} is a free parameter.

For the resonances one gets

$$r = -1, r = 0, r = 1(\text{threefold}), r = 2, r = 4(\text{twofold}). \quad (4.11)$$

Since a_{40} or a_{60} is arbitrary in our first step of the singular point analysis, we find that $r = 0$ is a resonance. To examine the behaviour in the neighbourhood of the singularity at z , one makes the ansatz

$$w_i(z) = \tau^{-1} \sum_{j=0}^{\infty} \tau^j a_{ij} \quad i = 1, 2, 3, 8$$

and

$$w_i(z) = \tau^{-2} \sum_{j=0}^{\infty} \tau^j a_{ij} \quad i = 4, 5, 6, 7.$$

On substitution of these Laurent expansions into (4.9) one obtains a system of linear equations. At the threefold resonance $j = 1$ the expansion coefficients a_{i1} ($i = 1, 2, \dots, 8$) are determined by the following system:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & m_{23} & 0 & m_{13} & 0 \\ 0 & 0 & 0 & -m_{23} & 0 & m_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{13} & 0 & -m_{23} & 0 \\ 0 & 0 & 0 & -1 & -\alpha a_{80} & 0 & 0 & -\alpha a_{50} \\ 0 & 0 & 0 & \alpha a_{80} & -1 & 0 & 0 & \alpha a_{40} \\ 0 & 0 & 0 & 0 & 0 & -1 & -\alpha a_{80} & -\alpha a_{70} \\ 0 & 0 & 0 & 0 & 0 & \alpha a_{80} & -1 & \alpha a_{60} \\ 0 & 0 & 0 & 0 & 3m_{31} & 0 & 3m_{23} & 0 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \\ a_{51} \\ a_{81} \\ a_{71} \\ a_{81} \end{bmatrix} = \begin{bmatrix} \beta_1 a_{20} \\ 2m_{12}a_{30} - \beta_1 a_{10} \\ -2m_{12}a_{20} \\ \beta_2 a_{50} + m_{12}a_{70} \\ -\beta_2 a_{40} - m_{12}a_{60} \\ \beta_3 a_{70} + m_{12}a_{50} \\ -\beta_3 a_{60} - m_{12}a_{40} \\ 0 \end{bmatrix}$$

$$\beta_1 = \omega_{02} - \omega_{01} \quad \beta_2 = \omega_{03} - \omega_{01} + \alpha N \quad \beta_3 = \omega_{03} - \omega_{02} + \alpha N \quad (4.12)$$

with $\alpha = \frac{1}{3}\gamma_3$.

For all values of the parameters m_{ij} and β_i this system of linear equations has no solution. The rank is equal to five and (4.12) can be solved if and only if $m_{12} = 0$ and $\beta_1 = 0$, i.e. $\omega_{01} = \omega_{02}$. Then the system admits a three-parameter solution:

$$a_{41} = a_{51} = a_{61} = a_{71} = 0 \quad a_{81} = -(\beta/\alpha) \quad \beta = \omega_{03} - \omega_{02} + \alpha N$$

with a_{11} , a_{21} and a_{31} arbitrary.

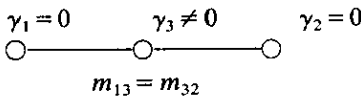
For $j \geq 2$ we obtain the recursion formula

$$\begin{bmatrix}
 j-1 & 0 & 0 & 0 & m_{23} & 0 & m_{13} & 0 \\
 0 & j-1 & 0 & -m_{23} & 0 & m_{13} & 0 & 0 \\
 0 & 0 & j-1 & 0 & m_{13} & 0 & -m_{23} & 0 \\
 0 & 0 & 0 & j-2 & -\alpha a_{80} & 0 & 0 & -\alpha a_{50} \\
 0 & 0 & 0 & \alpha a_{80} & j-2 & 0 & 0 & \alpha a_{40} \\
 0 & 0 & 0 & 0 & 0 & j-2 & -\alpha a_{80} & -\alpha a_{70} \\
 0 & 0 & 0 & 0 & 0 & \alpha a_{80} & j-2 & \alpha a_{60} \\
 0 & 0 & 0 & 0 & 3m_{31} & 0 & 3m_{32} & j-1
 \end{bmatrix}
 \begin{bmatrix}
 a_{1j} \\
 a_{2j} \\
 a_{3j} \\
 a_{4j} \\
 a_{5j} \\
 a_{6j} \\
 a_{7j} \\
 a_{8j}
 \end{bmatrix}
 =
 \begin{bmatrix}
 0 \\
 0 \\
 0 \\
 \beta a_{5j-1} - m_{32} a_{2j-2} + \alpha \sum_{k=1}^{j-1} a_{5k} a_{8j-k} \\
 -\beta a_{4j-1} + m_{13}(a_{3j-2} + a_{8j-2}) + m_{23} a_{1j-2} - \alpha \sum_{k=1}^{j-1} a_{4k} a_{8j-k} \\
 \beta a_{7j-1} + m_{31} a_{2j-2} + \alpha \sum_{k=1}^{j-1} a_{7k} a_{8j-k} \\
 -\beta a_{6j-1} + m_{31} a_{1j-2} - m_{32}(a_{3j-2} - a_{8j-2}) - \alpha \sum_{k=1}^{j-1} a_{6k} a_{8j-k} \\
 0
 \end{bmatrix}. \tag{4.13}$$

At the resonance $j=2$ we find that a_{82} can be chosen arbitrary provided the compatibility condition $m_{13} = m_{23}$ is satisfied, which leads to a further restriction on the intersite matrix elements.

At the twofold resonance $j = 4$ we get from the recursion formula (4.13) a set of equations where both the fourth and the fifth equations and the sixth and the seventh equations are equal. Therefore, six equations are left for the eight coefficients a_{i4} ($i = 1, 2, \dots, 8$). That means two expansion coefficients are arbitrary. Together with the pole position at $z = z_1$, the eight constants of integration are obtained. The Laurent expansion exists and system (4.9) has the Painlevé property.

One finds that in the case of a single-site anharmonicity the configuration of an open chain with identical intersite matrix elements, responsible for the linear interaction of the only anharmonic site with its neighbouring sites, is integrable, i.e.



with regard to the invariance to any site index permutation.

The trimer results of this section yield that this is the only integrable configuration for the DST system with $n = 3$: all other configurations are non-integrable. Finally, we comment on the possibility that $\bar{\gamma}$ could vanish even for non-vanishing γ_i s. Setting $\bar{\gamma} = \gamma_1 \gamma_2 + \gamma_1 \gamma_3 + \gamma_2 \gamma_3 = 0$, one obtains

$$\gamma_1 = -\gamma_2 \gamma_3 / (\gamma_2 + \gamma_3). \tag{4.14}$$

Even though this is unphysical because at least one γ_i has to be negative it is still interesting from a purely analytical point of view.

The resonance analysis may be repeated and yields that three of the resonances are irrational and, hence, the system in this case is not of the Painlevé type.

5. Summary

We have discussed the DST equation, including disorder, in all system parameters for three degrees of freedom, from which we derived a system of eight real first-order ODEs. We found that the $n = 3$ DST equation possesses the Painlevé property only in the case of a single-site anharmonicity for the configuration of an open chain; this means that the system of ODEs is integrable in this special case only. In all other cases, i.e. for two- and three-site anharmonicities, we find that the DST equation is non-integrable. We note that this behaviour is present for both the symmetric (equal γ_i s) and the asymmetric (unequal γ_i s) cases of site anharmonicities, i.e. disorder in the nonlinearity parameter is not crucial in causing non-integrability.

The extension to larger DST systems with $n > 3$ should certainly be of interest.

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